

# Conservative interacting particles system with anomalous rate of ergodicity. \*

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## Abstract

We analyze certain conservative interacting particle system and establish ergodicity of the system for a family of invariant measures. Furthermore, we show that convergence rate to equilibrium is exponential. This result is of interest because it presents counterexample to the standard assumption of physicists that conservative system implies polynomial rate of convergence.

Keywords: Hörmander type generators, conservative interacting particle system, ergodicity.

## 1 Introduction

In this paper we present an example of the conservative interacting particle system with exponential rate of convergence to equilibrium. This system naturally appears in the dyadic model of turbulence (see [2]). In [2] it has been established that the system has anomalous dissipation. This result seems to be the reason behind exponential rate of convergence to equilibrium. Similar systems naturally appear in the models of heat conduction and quantum spin chains ([4],[5],[11],[12],[13]).

Ergodic properties of systems of interacting particles is one of the central topics of statistical mechanics. They have been studied starting from the works of Spitzer [22] and Dobrushin [9]. The literature of the subject is huge and we will not attempt to list it here, see [20] and references therein.

Interacting particle systems are usually divided into two classes: conservative and nonconservative ones. Conservative ones are presumed to have at most polynomial rate of convergence to equilibrium and dissipative ones exponential one ([19]).

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In the same time rigorous mathematical results about rates of convergence to equilibrium of conservative systems has been established only in the handful of cases such as Kawasaki dynamics ([6],[7]), Ginzburg-Landau type processes ([16],[19]) and Brownian moment processes ([15]). The result of this paper shows that existence of formal conservation law does not necessarily imply polynomial rate of convergence. Consequently, "meta" theorem that conservative interacting particle systems are ergodic with polynomial rate of convergence to equilibrium is not correct.

## 2 The system

Let  $(\Omega, F_t, P)$  be a filtered probability space and  $(W_n)$  be a sequence of independent Brownian motions. Consider the equation

$$dX_n = k_{n-1}X_{n-1} \circ dW_{n-1} - k_n X_{n+1} \circ dW_n, \quad X_n(0) = X_n^{(0)} \quad (2.1)$$

for all  $n \geq 1$ , with  $X_0 = 0$ ,  $k_0 = 0$ , and  $k_n = \lambda^n, n \in \mathbb{N}$  for some  $\lambda > 1$ ,  $X_n^{(0)}$  deterministic or  $F_0$ -adapted. The stochastic integral in the system (2.1) is in Stratonovich sense.

**Remark 2.1** *The assumption  $k_n = \lambda^n, n \in \mathbb{N}$  has been imposed for simplicity. It can be relaxed to the assumption that the sequence  $\left\{\frac{k_{n+1}}{k_n}\right\}_{n=1}^{\infty}$  is nondecreasing and the first term of the sequence is bigger than 1.*

Consider the space

$$W = \left\{ (x_n)_{n \in \mathbb{N}} : \|x\|_W^2 := \sum k_n^{-2} x_n^2 < \infty \right\}.$$

**Definition 2.2** *We say that a sequence of continuous adapted processes  $(X_n)$  is a weak (in the analytical sense) solution in  $W$  of equation (2.1) if  $(X_n)$  is  $L^\infty([0, T]; L^2(\Omega; W))$  and*

$$dX_n = k_{n-1}X_{n-1}dW_{n-1} - k_n X_{n+1}dW_n - \frac{1}{2} (k_n^2 + k_{n-1}^2) X_n dt$$

for each  $n \geq 1$ . If we have

$$E \left[ \|X(t)\|_W^2 \right] \leq E \left[ \|X^{(0)}\|_W^2 \right], \quad a.e. \ t \geq 0$$

we say it is a Leray solution in  $W$ .

**Theorem 2.3** *For every  $X^{(0)} \in L^2(\Omega; W)$ ,  $F_0$ -measurable, there exists a weak Leray solution in  $W$  of equation (2.1).*

**Remark 2.4** *We use Galerkin type finite dimensional approximation to show existence of solution of system (2.1). Different way would be to apply results of Holevo [14].*

**Proof. Step 1** (existence). For each  $N \in \mathbb{N}$ , consider the finite dimensional system

$$dX_n^N = k_{n-1}X_{n-1}^N dW_{n-1} - k_n X_{n+1}^N dW_n - \frac{1}{2} (k_n^2 + k_{n-1}^2) X_n^N dt, \quad n = 1, \dots, N$$

with  $X_0^N = X_{N+1}^N = 0$  and the initial condition  $X_n^N(0)$  equal to  $X_n^{(0)}$ ,  $n = 1, \dots, N$ . This system has a unique strong solution, with all moments finite. Indeed, it immediately follows from the Theorem 3.3, p. 7 of [3]. Set

$$q_n^N = E \left[ (X_n^N)^2 \right].$$

By Itô formula (we need finite fourth moments to have that the Itô terms are true martingales, then they disappear taking expected value) we have (we drop  $N$ )

$$\begin{aligned} q_n' &= - (k_n^2 + k_{n-1}^2) q_n + k_{n-1}^2 q_{n-1} + k_n^2 q_{n+1} \\ &= -k_{n-1}^2 (q_n - q_{n-1}) + k_n^2 (q_{n+1} - q_n) \end{aligned}$$

for  $n = 1, \dots, N$ , with  $q_0 = q_{N+1} = 0$ . Denote by  $\|\cdot\|_W^2$  the same norm introduced above also in the case of a finite number of components. We have

$$\begin{aligned} \frac{d}{dt} E \left[ \|X^N\|_W^2 \right] &= \sum_{n=1}^N k_n^{-2} \frac{d}{dt} q_n^N = \\ &= - \sum_{n=1}^N k_n^{-2} k_{n-1}^2 (q_n - q_{n-1}) + \sum_{n=1}^N k_n^{-2} k_n^2 (q_{n+1} - q_n) \\ &= -\lambda^{-2} \sum_{n=2}^N (q_n - q_{n-1}) + \sum_{n=1}^N (q_{n+1} - q_n) \leq -q_1. \end{aligned}$$

Since  $q_1 \geq 0$  by definition, we have

$$E \left[ \|X^N(t)\|_W^2 \right] \leq E \left[ \|X^N(0)\|_W^2 \right], \quad t \geq 0. \quad (2.2)$$

Thus the sequence  $(X^N)_{N \geq 0}$  is bounded in  $L^\infty([0, T]; L^2(\Omega; W))$ . Therefore, there exists a subsequence  $N_k \rightarrow \infty$  such that  $(X_n^{(N_k)})_{n \geq 1}$  converges weakly to some  $(X_n)_{n \geq 1}$  in  $L^2(\Omega \times [0, T]; W)$  and also weak star in  $L^\infty([0, T]; L^2(\Omega; W))$ . Now the proof proceeds by standard arguments typical of equations with monotone operators (which thus apply to linear equations), presented in [21], [18]. The subspace of  $L^2(\Omega \times [0, T]; W)$  of progressively measurable processes is strongly closed, hence weakly closed, hence  $(X_n)_{n \geq 1}$  is progressively measurable. The one-dimensional stochastic integrals which appear in each equation of the system are (strongly) continuous linear operators from the subspace of  $L^2(\Omega \times [0, T])$  of progressively measurable processes to  $L^2(\Omega)$ , hence they are

weakly continuous, a fact that allows us to pass to the limit in each one of the linear equations of the system. A posteriori, from these integral equations, it follows that there is a modification such that all components are continuous. The proof of existence is complete.

**Remark 2.5** *If initial condition  $X_0 \in L^2(\Omega, l^2)$  then there exists unique solution  $X \in L^\infty([0, T]; L^2(\Omega; l^2))$  of equation (2.1) (see Theorem 3.3 in [3]).*

**Step 2** (Leray solution). From (2.2) and the definition of  $X^N(0)$  we have

$$E \left[ \|X^N(t)\|_W^2 \right] \leq E \left[ \|X^{(0)}\|_W^2 \right], \quad t \geq 0.$$

Hence

$$E \left[ \int_a^b \|X^N(t)\|_W^2 dt \right] \leq (b-a) E \left[ \|X^{(0)}\|_W^2 \right], \quad 0 \leq a \leq b.$$

Weak convergence in  $L^2(\Omega \times [0, T]; W)$  implies that

$$E \left[ \int_a^b \|X(t)\|_W^2 dt \right] \leq (b-a) E \left[ \|X^{(0)}\|_W^2 \right], \quad 0 \leq a \leq b.$$

By Lebesgue differentiation theorem, we get  $E \left[ \|X(t)\|_W^2 \right] \leq E \left[ \|X^{(0)}\|_W^2 \right]$  for a.e.  $t$ .

■

**Remark 2.6** *The question of uniqueness of weak Leray solutions of system (2.1) is an open problem. The difficulty lies in showing that the difference of two weak non trivial<sup>1</sup> Leray solutions  $X^1$  and  $X^2$  is a weak Leray solution itself. It seems that this difficulty is related to the fact that the closed infinite system of linear equations for a sequence  $q_n(t) = \mathbb{E} [|X_n^1(t) - X_n^2(t)|^2], n = 1, \dots$ :*

$$q'_n = -k_{n-1}^2 (q_n - q_{n-1}) + k_n^2 (q_{n+1} - q_n) \quad (2.3)$$

*has non-unique solutions in a positive cone of a Banach space space  $l_k^1 = \{x \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} \frac{|x_n|}{k_n^2} < \infty\}$  endowed with a norm  $|x|_{l_k^1} = \sum_{n=1}^{\infty} \frac{|x_n|}{k_n^2}$ . Indeed, denote  $c = 2 \sup_{t \geq 0} \mathbb{E} [\|X^1(t)\|_W^2 + \|X^2(t)\|_W^2]$ . Notice that  $0 < c < \infty$  because  $X^1$  and  $X^2$  are non identically zero Leray solutions of the system (2.1). Define  $p_n = \frac{q_n}{k_n^2 c}, n \in \mathbb{N}, p = (p_n)_{n=1}^{\infty}$ . Then from the identity*

$$\sum_{n=1}^{\infty} p_n(t) = \frac{\mathbb{E} \|X^1(t) - X^2(t)\|_W^2}{2 \sup_{s \geq 0} \mathbb{E} [\|X^1(s)\|_W^2 + \|X^2(s)\|_W^2]}, \quad (2.4)$$

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<sup>1</sup>non identically zero and non proportional to each other

we can deduce that

$$\sum_{n=1}^{\infty} p_n(t) \leq 1, t \geq 0. \quad (2.5)$$

Moreover,

$$\exists t_0 \geq 0 \text{ such that } \sum_{n=1}^{\infty} p_n(t_0) < 1. \quad (2.6)$$

Indeed, otherwise it follows from the identity (2.4) that  $X^2 = -X^1$  a.s. Furthermore, because  $k_n = \lambda^n$ , we have

$$\frac{d}{dt}p = pA, \quad (2.7)$$

$$A = \begin{pmatrix} -k_1^2 & 1 & 0 & 0 & \dots & \dots & \dots & \dots \\ k_2^2 & -(k_1^2 + k_2^2) & k_1^2 & 0 & \dots & \dots & \dots & \dots \\ 0 & k_3^2 & -(k_2^2 + k_3^2) & k_2^2 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 0 & k_n^2 & -(k_n^2 + k_{n-1}^2) & k_{n-1}^2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Matrix  $A$  has tridiagonal form with positive off diagonal entries. Consequently, we can construct birth and death process  $\xi_t, t \geq 0$  on some new probability space  $(S, \mathcal{G}, \mathbb{P}')$  such that matrix  $A$  is a  $q$ -matrix of the process (see details of construction at p. 8-10 [2]) and  $p_n(t) = \mathbb{P}'(\xi_t = n), n \in \mathbb{N}$ . Now the question of uniqueness of the solution of the system (2.7) in conjunction with conditions (2.5) and (2.6) can be reformulated as the question of uniqueness (in law) of the process  $\xi_t, t \geq 0$  with given  $q$ -matrix  $A$ . Hence the non uniqueness of the solution of the system (2.7) with conditions (2.5) and (2.6) follow for instance from the criterion in Theorem 3.2.3, p. 101 of monograph [1] (see also the original paper by Kato [17]). Indeed, we have that the criterion number  $S$  is less than infinity for the  $q$ -matrix  $A$  defined above and condition (2.6) means that the solution is dishonest<sup>2</sup>.

Now we will show existence and uniqueness of solution of system (2.1) in a more restrictive class of *moderate* solutions. We will need following Lemma.

**Lemma 2.7** *If  $X(0) \in L^2(\Omega, l^2)$  then the unique weak solution  $X \in L^\infty([0, T]; L^2(\Omega; l^2))$  of the system (2.1) (constructed in Theorem 3.3 in [3]) is a Leray solution i.e.*

$$E \left[ \|X(t)\|_W^2 \right] \leq E \left[ \|X^{(0)}\|_W^2 \right], \quad a.e. t \geq 0$$

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<sup>2</sup>i.e.  $\mathbb{P}'$ -a.s. process  $\xi$  escapes to infinity in finite time

**Proof.** We have

$$\begin{aligned}
& \frac{d}{dt} \sum_{n=1}^N \frac{1}{k_n^2} \mathbb{E} [|X_n(t)|^2] \\
&= -\lambda^{-2} \sum_{n=2}^N (\mathbb{E} [|X_n(t)|^2] - \mathbb{E} [|X_{n-1}(t)|^2]) + \sum_{n=1}^N (\mathbb{E} [|X_{n+1}(t)|^2] - \mathbb{E} [|X_n(t)|^2]) \\
&= \mathbb{E} [|X_{N+1}(t)|^2] + (\lambda^{-2} - 1) \mathbb{E} [|X_1(t)|^2] - \lambda^2 \mathbb{E} [|X_N(t)|^2] \\
&\leq \mathbb{E} [|X_{N+1}(t)|^2], \quad N \in \mathbb{N}.
\end{aligned}$$

Consequently, we have

$$\sum_{n=1}^N \frac{1}{k_n^2} \mathbb{E} [|X_n(t)|^2] \leq \sum_{n=1}^N \frac{1}{k_n^2} \mathbb{E} [|X_n(0)|^2] + \int_0^t \mathbb{E} [|X_{N+1}(s)|^2] \, ds.$$

Moreover, by  $X \in L^\infty([0, T]; L^2(\Omega; l^2))$  we have

$$\lim_{N \rightarrow \infty} \int_0^t \mathbb{E} [|X_{N+1}(s)|^2] \, ds = 0,$$

and the result follows. ■

**Corollary 2.8** Assume that the sequence  $\{X^{(N)}(0)\}_{N \in \mathbb{N}} \subset L^2(\Omega, l^2)$  of  $\mathcal{F}_0$ -measurable functions satisfy

$$(L^2(\Omega, W)) - \lim_{N \rightarrow \infty} X^{(N)}(0) = X(0) \in L^2(\Omega, W).$$

Then the sequence  $\{X^{(N)}\}_{N \in \mathbb{N}}$  is a Cauchy sequence in  $L^\infty([0, T], L^2(\Omega, W))$  and, consequently, there exist  $X \in L^\infty([0, T], L^2(\Omega, W))$  such that

$$X = (L^\infty([0, T], L^2(\Omega, W))) - \lim_{N \rightarrow \infty} X^{(N)}.$$

**Corollary 2.9** Assume that the sequences  $\{\tilde{X}^{(N)}(0)\}_{N \in \mathbb{N}}, \{X^{(N)}(0)\}_{N \in \mathbb{N}} \subset L^2(\Omega, l^2)$  of  $\mathcal{F}_0$ -measurable functions satisfy

$$(L^2(\Omega, W)) - \lim_{N \rightarrow \infty} X^{(N)}(0) = X(0) \in L^2(\Omega, W),$$

$$(L^2(\Omega, W)) - \lim_{N \rightarrow \infty} \tilde{X}^{(N)}(0) = \tilde{X}(0) \in L^2(\Omega, W),$$

and

$$X = (L^\infty([0, T], L^2(\Omega, W))) - \lim_{N \rightarrow \infty} X^{(N)}, \quad \tilde{X} = (L^\infty([0, T], L^2(\Omega, W))) - \lim_{N \rightarrow \infty} \tilde{X}^{(N)}.$$

Then

$$\mathbb{E} [\|X(t) - \tilde{X}(t)\|_W^2] \leq \mathbb{E} [\|X(0) - \tilde{X}(0)\|_W^2].$$

**Proof.** We have by triangle inequality and inequality  $(a+b)^2 \leq (1+\epsilon)a^2 + (1+\frac{1}{\epsilon})b^2$ ,  $a, b \in \mathbb{R}$ ,  $\epsilon > 0$  that for any  $\epsilon > 0$  there exists  $C(\epsilon) > 0$  such that

$$\begin{aligned} \mathbb{E} \left[ \|X(t) - \tilde{X}(t)\|_W^2 \right] &\leq C(\epsilon) \mathbb{E} \left[ \|X(t) - X^{(N)}(t)\|_W^2 \right] \\ &\quad + C(\epsilon) \mathbb{E} \left[ \|\tilde{X}(t) - \tilde{X}^{(N)}(t)\|_W^2 \right] \\ &\quad + (1+\epsilon) \mathbb{E} \left[ \|X^{(N)}(t) - \tilde{X}^{(N)}(t)\|_W^2 \right]. \end{aligned}$$

Taking limit  $N \rightarrow \infty$  we conclude that

$$\begin{aligned} \mathbb{E} \left[ \|X(t) - \tilde{X}(t)\|_W^2 \right] &\leq (1+\epsilon) \lim_{N \rightarrow \infty} \mathbb{E} \left[ \|X^{(N)}(t) - \tilde{X}^{(N)}(t)\|_W^2 \right] \\ &\leq (1+\epsilon) \lim_{N \rightarrow \infty} \mathbb{E} \left[ \|X^{(N)}(0) - \tilde{X}^{(N)}(0)\|_W^2 \right] \\ &\leq (1+\epsilon) \mathbb{E} \left[ \|X(0) - \tilde{X}(0)\|_W^2 \right], \end{aligned}$$

and the result follows. ■

Now we are ready to define class of moderate solutions.

**Definition 2.10** We call  $X \in L^\infty([0, T], L^2(\Omega, W))$  with initial condition  $X(0) \in L^2(\Omega, W)$  a moderate solution of the system (2.1) iff  $X$  is a weak Leray solution and there exists sequence  $X^{(N)} \in L^\infty(\Omega \times [0, T], l_2)$  of weak solutions of system (2.1) with  $X^{(N)}(0) \in L^2(\Omega, l^2)$  such that

$$X = (L^\infty([0, T], L^2(\Omega, W))) - \lim_{N \rightarrow \infty} X^{(N)}.$$

**Theorem 2.11** Given  $X(0) \in L^2(\Omega, W)$  there exists a unique moderate solution of the system (2.1). Furthermore, it depends continuously on its initial condition in the following sense. If  $X^\eta$  and  $X^\rho$  are the solutions corresponding to the initial conditions  $\eta, \rho \in L^2(\Omega; W)$ ,  $F_0$ -measurable, then:

i)

$$E \left[ \|X^\eta(t) - X^\rho(t)\|_W^2 \right] \leq E \left[ \|\eta - \rho\|_W^2 \right], \quad \text{a.e. } t \geq 0$$

ii) for every  $N > 0$

$$E \left[ \sum_{n=1}^N k_n^{-2} (X_n^\eta(t) - X_n^\rho(t))^2 \right] \leq E \left[ \|\eta - \rho\|_W^2 \right], \quad \text{for all } t \geq 0.$$

**Proof.** Existence has been shown in the Corollary (2.8). Uniqueness and (i) follow from the Corollary (2.9). Then (ii) holds by continuity of single components and Fatou theorem. ■

**Remark 2.12** The same arguments show that the weak Leray solution constructed in the Theorem (2.3) is a moderate solution.

### 3 Markov property

We have proved that, for every  $x \in W$  there is a unique moderate solution  $(X_n^x(t))$  in  $W$ . Let us prove that the family  $X^x$  is a Markov process.

**Definition 3.1** Define the operator  $P_t$  on  $B_b(W)$  as

$$(P_t \varphi)(x) := E[\varphi(X^x(t))].$$

By the previous result,  $P_t$  is well defined also from  $C_b(W)$  to  $C_b(W)$ .

**Proposition 3.2** We have

$$E[\varphi(X^x(t+s)) | F_t] = (P_s \varphi)(X^x(t)) \quad (3.1)$$

for all  $\varphi \in C_b(W)$ , hence the family  $X^x$  is a Markov process. The Markov semigroup  $P_t$  is Feller.

**Proof.** We have just to prove the identity (3.1), the other claims being obvious or classical. Indeed, Feller property follows from part i) of Theorem 2.11. It is enough to show identity (3.1) when  $x \in l^2$ . Indeed, general case will follow from the continuous dependence of moderate solution upon initial condition (part i) of Theorem 2.11).

Consider the equation on a generic interval  $[s, t]$  with initial condition  $\eta \in L^2(\Omega; l^2)$ ,  $F_s$ -measurable, at time  $s$  and call  $X^{s, \eta}(t)$  the solution. Consider the function

$$Y(t) := \begin{cases} X^x(t) & \text{for } t \in [0, s] \\ X^{s, X^x(s)}(t) & \text{for } t \geq s. \end{cases}$$

Direct substitution into the equations prove that  $Y$  is a solution with initial condition  $x$ , hence equal to  $X^x(t)$  also for  $t \geq s$ . This proves the evolution property

$$X^{s, X^x(s)}(t) = X^x(t), \quad t \geq s.$$

Thus

$$E[\varphi(X^x(t+s)) | F_t] = E\left[\varphi\left(X^{t, X^x(t)}(t+s)\right) | F_t\right].$$

If we prove that

$$E[\varphi(X^{t, \eta}(t+s)) | F_t] = (P_s \varphi)(\eta)$$

for all  $\eta \in L^2(\Omega; l^2)$ ,  $F_t$ -measurable, we are done. If  $\eta = x$ , a.s. constant, it is true, by exploiting the fact that the increments of the Brownian motions  $W_n$  from  $t$  to  $t+s$  are independent of  $F_t$ ; and because the dynamics is autonomous. From constant values one generalizes to  $\eta = \sum_{i=1}^n x_i 1_{A_i}$ ,  $A_i \in F_t$ ; indeed, for such  $\eta$ , we have

$$X^{t, \eta}(t+s) = \sum_{i=1}^n X^{t, x_i}(t+s) 1_{A_i}.$$

Finally we have the identity for all  $\eta$  by the continuity result above. ■



## 4 Invariant measures

Consider the measures  $\mu_r$ , parametrized by  $r \geq 0$ , formally defined as

$$\mu_r(dx) = \frac{1}{Z} \exp\left(-\frac{\sum_{n=1}^{\infty} x_n^2}{2r}\right) dx.$$

The rigorous definition is:  $\mu_r$  is the Gauss measure on  $l^2$ , namely the Gaussian measure on  $W$  having covariance equal to identity. For every function  $f$  of the first  $n$  coordinates only of  $l^2$ , the measure  $\mu_r$  is given by

$$\int_Y f(x_1, \dots, x_n) \mu_r(dx) = \frac{1}{Z_n} \int_{\mathbb{R}^n} f(x_1, \dots, x_n) \exp\left(-\frac{\sum_{k=1}^n x_k^2}{2r}\right) dx_1 \dots dx_n$$

where  $Z_n = (2\pi r)^{n/2}$ . This formula identifies  $\mu_r$ .

Moreover, for technical reasons, we need

$$\widetilde{W} = \left\{ (x_n)_{n \in \mathbb{N}} : \|x\|_W^2 := \sum k_n^{-4} x_n^2 < \infty \right\}.$$

Notice that

$$\mu_r(\widetilde{W}) = 1$$

and the embedding of  $W$  in  $\widetilde{W}$  is compact.

**Proposition 4.1** *For every  $r > 0$ ,  $\mu_r$  is invariant for the Markov semigroup  $P_t$  defined above on  $W$ .*

**Proof.** It is sufficient to prove

$$\int_Y (P_t \varphi)(x) \mu_r(dx) = \int_Y \varphi(x) \mu_r(dx)$$

for all  $\varphi$  of the form  $\varphi(x) = f(x_1, \dots, x_n)$ , with bounded continuous  $f$ . We have

$$\int_Y (P_t \varphi)(x) \mu_r(dx) = E \left[ \int_Y f(X_1^x(t), \dots, X_n^x(t)) \mu_r(dx) \right].$$

The strategy now is the following one. On an enlarged probability space, if necessary, we can define an  $F_0$ -measurable r.v.  $X^{(0)} \in L^2(\Omega; W)$  with law  $\mu_r$ . For every  $N > 0$  denote by  $X^N$  the Galerkin approximations used to prove existence above, with initial condition  $X_n^N(0) = X_n^{(0)}$ . Sequence  $X^N$  weakly converges to the moderate solution  $X$  having initial condition  $X^{(0)}$ . Denote by  $\rho_N$  the law of  $X^N$  and by  $\rho$  the law of  $X$ , on  $L^2([0, T]; W)$ . We shall prove that the sequence  $\rho_N$  is tight in  $L^2([0, T]; \widetilde{W})$ . Then there exists a subsequence  $\rho_{n_k}$  weakly convergent to some probability measure on  $L^2([0, T]; \widetilde{W})$ . Such measure must be  $\rho$ .

It is enough to show that the sequence  $X^N$  of Galerkin approximations is bounded in  $L^2(\Omega, L^2([0, T], W)) \cap L^2(\Omega, W^{\alpha, 2}([0, T], \widetilde{W}))$ ,  $\alpha \in (0, 1)$ . That implies that laws  $\{\rho_N\}_{N=1}^\infty$  are bounded in probability on

$$L^2([0, T], W) \cap W^{\alpha, 2}([0, T], \widetilde{W}), \alpha \in (0, 1).$$

Since embedding

$$L^2([0, T], W) \cap W^{\alpha, 2}([0, T], \widetilde{W}) \subset L^2([0, T], \widetilde{W}), \alpha \in (0, 1).$$

is compact by Theorem 2.1, p. 370 of [10] (applied with  $B_0 = W$ ,  $B = B_1 = \widetilde{W}$ ,  $p = 2$ ) we shall conclude that the sequence  $\rho_N$  is tight in  $L^2([0, T]; \widetilde{W})$ .

Since the sequence  $(X^N)$  is bounded in  $L^\infty([0, T], L^2(\Omega, W))$  it remains to show that the sequence  $(X^N)$  is bounded in  $L^2(\Omega, W^{\alpha, 2}([0, T], \widetilde{W}))$  for some  $\alpha \in (0, 1)$ .

Decompose  $X^N$  as

$$X^N(t) = X^N(0) - \int_0^t A^N X^N(s) ds + \int_0^t B^N(X^N) dW(s) = J_1^N(t) + J_2^N(t) + J_3^N(t)$$

where

$$\begin{aligned} (A^N x)_{n,m} &= -\frac{\delta_{n,m}}{2} (k_{n-1}^2 + k_n^2) x_n, \\ (B^N x)_{n,m} &= k_{n-1} x_{n-1} \delta_{n,m+1} - k_n x_{n+1} \delta_{n,m}, x \in P_N(W), m, n = 1, \dots, N. \end{aligned}$$

We have

$$\mathbb{E}|J_1^N|_{W^{1,2}(0,T;W)}^2 \leq T \mathbb{E}|X^{(0)}|_W^2. \quad (4.1)$$

Since  $|A^N|_{\mathcal{L}(W, \widetilde{W})} \leq K = 1 + \sup_n \frac{k_{n-1}^2}{k_n^2}$  we infer that

$$\mathbb{E}|J_2^N|_{W^{1,2}(0,T;\widetilde{W})}^2 \leq C(T, K) \mathbb{E}|X^N|_{L^2([0,T],W)}^2 \leq C(T, K) \mathbb{E}|X^{(0)}|_W^2. \quad (4.2)$$

Fix  $\alpha \in (0, \frac{1}{2})$ . By Lemma 2.1, p. 369 of [10] we have that

$$\mathbb{E}|J_2^N|_{W^{\alpha,2}(0,T;\widetilde{W})}^2 \leq \mathbb{E} \int_0^T |B^N(X^N)|_{L_{HS}(l^2, \widetilde{W})}^2 ds \quad (4.3)$$

Notice that

$$\begin{aligned} |B(x)|_{L_{HS}(l^2, \widetilde{W})}^2 &= \sum_{n=1}^\infty |B(x)(e_n)|_{\widetilde{W}}^2 \\ &\leq \sum_{n=1}^\infty k_n^{-4} k_n^2 x_{n+1}^2 + k_{n+1}^{-4} k_n^2 x_n^2 \leq (K + K^2) |x|_W^2, x \in W. \end{aligned} \quad (4.4)$$

where  $(e_n)_{n=1}^\infty$  is ONB in  $l^2$ .

Combining inequalities (4.3) and (4.4) we infer that

$$\mathbb{E}|J_2^N|^2_{W^{\alpha,2}(0,T;\widetilde{W})} \leq C\mathbb{E} \int_0^T |X^N(s)|_W^2 ds \leq C(T, K, \alpha)\mathbb{E}|X^{(0)}|_W^2. \quad (4.5)$$

Hence, inequalities (4.1), (4.2) and (4.5) imply that for some  $\alpha \in (0, \frac{1}{2})$

$$\mathbb{E}|X^N|^2_{W^{\alpha,2}([0,T],\widetilde{W})} \leq C(T, K, \alpha)\mathbb{E}|X^{(0)}|_W^2, \quad (4.6)$$

and the result follows. ■

**Corollary 4.2** *The semigroup  $(P_t)_{t \geq 0}$  acting on  $C_b(W)$  can be extended to  $L^p(W, \mu_r)$  for any  $p \geq 1$ . Generator of the semigroup  $(P_t)_{t \geq 0}$  is given by the formula*

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^{\infty} k_j^2 D_{j,j+1}^2$$

with  $D_{j,j+1} = x_j \partial_{x_{j+1}} - x_{j+1} \partial_{x_j}$ ,  $j \in \mathbb{N}$ .

**Proof.** It follows from Itô formula. ■

## 5 Symmetry of the generator in the Sobolev spaces

Let

$$L = \sum_{i=1}^{\infty} \left( \frac{\partial^2}{\partial x_i^2} - x_i \frac{\partial}{\partial x_i} \right)$$

be Ornstein-Uhlenbeck operator and

$$\mathcal{H}^n = \left\{ f \in L^2(W, \mu_r) : |f|_{\mathcal{H}^n}^2 = |f|_{L^2(W, \mu_r)}^2 + ((-L)^n f, f)_{L^2(W, \mu_r)} < \infty \right\}, n \in \mathbb{Z},$$

$$\mathcal{C} = \left\{ \phi : W \rightarrow \mathbb{R}, \phi(x) = f(x_1, \dots, x_n), f \in C^4(\mathbb{R}^n, \mathbb{R}), n \in \mathbb{N} \right\}.$$

We have

$$[D_{i,i+1}, L]\phi = 0, \phi \in \mathcal{C}.$$

Consequently,

$$[\mathcal{L}, L]\phi = 0, \phi \in \mathcal{C},$$

and

**Proposition 5.1** *For all  $f, g \in \mathcal{C}$  we have*

$$(f, \mathcal{L}g)_{\mathcal{H}^n} = (g, \mathcal{L}f)_{\mathcal{H}^n} = - \sum_{l=1}^{\infty} k_l^2 (D_{l,l+1} f, D_{l,l+1} g)_{\mathcal{H}^n}, n \in \mathbb{Z}.$$

Fix  $n \in \mathbb{N} \cup 0$ .

**Corollary 5.2** *The operator  $\mathcal{L}$  is closable in  $\mathcal{H}^n$  and its closure has bounded from above self-adjoint extension, which we continue to denote by the same symbol  $\mathcal{L}$ . Moreover, the self-adjoint extension  $\mathcal{L}$  generates a strongly continuous contraction semigroup  $T_t = e^{t\mathcal{L}} : \mathcal{H}^n \rightarrow \mathcal{H}^n$  such that  $T_t = P_t|_{\mathcal{H}^n}$ .*

## 6 Ergodicity

Define

$$\mathcal{A}_r(f) = \sum_{n=1}^{\infty} |\partial_n f|_{L^2(W, \mu_r)}^2 = (-Lf, f)_{L^2(W, \mu_r)}, \quad (6.1)$$

$$\nu = \sum_{n=1}^{\infty} \frac{n}{k_n^2}, \text{ (where } k_n = \lambda^n, \lambda > 1). \quad (6.2)$$

Let us remind the reader that  $P_t : L^2(W, \mu_r) \rightarrow L^2(W, \mu_r), t \geq 0$  is the semi-group with the infinitesimal generator

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^{\infty} k_j^2 D_{j,j+1}^2.$$

Its existence, construction and properties has been discussed above, see the definition 3.1, the proposition 3.2 and the corollaries 4.2 and 5.2.

**Theorem 6.1** *There exist  $C = C(\lambda) > 0$  such that for any  $f \in \mathcal{H}^1$  and  $t \geq 0$*

$$\mathcal{A}_r(P_t f) \leq C \mathcal{A}_r(f) e^{-\frac{t}{\nu}}, f \in \mathcal{H}^1. \quad (6.3)$$

**Proof.**

It is enough to show (6.3) for  $f \in C_b^4(W)$ . Indeed,  $C_b^4(W)$  is dense in  $\mathcal{H}^1$  and  $(P_t)_{t \geq 0}$  is a contraction on  $\mathcal{H}^1$  by 5.2.

Denote  $f_t = P_t f$  for  $t \geq 0$ . For  $i \in \mathbb{N}$ , we can calculate that

$$\begin{aligned} |\partial_i f_t|^2 - P_t |\partial_i f|^2 &= \int_0^t \frac{d}{ds} P_{t-s} |\partial_i f_s|^2 ds \\ &= \int_0^t P_{t-s} (-\mathcal{L}(|\partial_i f_s|^2) + 2\partial_i f_s \mathcal{L} \partial_i f_s + 2\partial_i f_s [\partial_i, \mathcal{L}] f_s) ds \\ &= \int_0^t P_{t-s} \left( -\sum_{m \in \mathbb{N}} k_m^2 |D_{m,m+1}(\partial_i f_s)|^2 \right. \\ &\quad \left. + \partial_i f_s (-(k_i^2 + k_{i-1}^2) \partial_i f_s + 2k_{i-1}^2 D_{i,i-1} \partial_{i-1} f_s + 2k_i^2 D_{i,i+1} \partial_{i+1} f_s) \right) ds. \end{aligned} \quad (6.4)$$

Integrating (6.4) with respect to the invariant measure  $\mu_r$  yields

$$\begin{aligned} \mu_r |\partial_i f_t|^2 - \mu_r |\partial_i f|^2 &= \int_0^t \left( -\sum_{m \in \mathbb{N}} k_m^2 |D_{m,m+1}(\partial_i f_s)|^2 \right. \\ &\quad \left. - (k_i^2 + k_{i-1}^2) \mu_r |\partial_i f_s|^2 + 2k_{i-1}^2 \mu_r (\partial_i f_s D_{i,i-1} \partial_{i-1} f_s) \right. \\ &\quad \left. + 2k_i^2 \mu_r (\partial_i f_s D_{i,i+1} \partial_{i+1} f_s) \right) ds. \end{aligned} \quad (6.5)$$

Notice that the operators  $D_{i,j}, i, j \in \mathbb{N}$ , are antisymmetric in  $L^2(\mu_r)$ . Therefore

$$\begin{aligned} \mu_r |\partial_i f_t|^2 - \mu_r |\partial_i f|^2 &= \int_0^t \left( - \sum_{m \in \mathbb{N}} k_m^2 \mu_r |D_{m,m+1}(\partial_i f_s)|^2 \right. \\ &\quad - (k_i^2 + k_{i-1}^2) \mu_r |\partial_i f_s|^2 - 2k_{i-1}^2 \mu_r (D_{i,i-1}(\partial_i f_s) \partial_{i-1} f_s) \\ &\quad \left. - 2k_i^2 \mu_r (D_{i,i+1}(\partial_i f_s) \partial_{i+1} f_s) \right) ds. \end{aligned} \quad (6.6)$$

Hence, by Young's inequality we deduce that

$$\begin{aligned} \mu_r |\partial_i f_t|^2 - \mu_r |\partial_i f|^2 &\leq \int_0^t \left( - \sum_{m \in \mathbb{N}} k_m^2 \mu_r |D_{m,m+1}(\partial_i f_s)|^2 \right. \\ &\quad - (k_i^2 + k_{i-1}^2) \mu_r |\partial_i f_s|^2 + k_{i-1}^2 \mu_r |D_{i,i-1} \partial_i f_s|^2 + k_{i-1}^2 \mu_r |\partial_{i-1} f_s|^2 \\ &\quad \left. + k_i^2 \mu_r |D_{i,i+1} \partial_i f_s|^2 + k_i^2 \mu_r |\partial_{i+1} f_s|^2 \right) ds \leq \\ &\leq \int_0^t \left( - \sum_{m \neq i, i-1} k_m^2 \mu_r |D_{m,m+1}(\partial_i f_s)|^2 \right. \\ &\quad \left. - (k_i^2 + k_{i-1}^2) \mu_r |\partial_i f_s|^2 + k_{i-1}^2 \mu_r |\partial_{i-1} f_s|^2 + k_i^2 \mu_r |\partial_{i+1} f_s|^2 \right) ds. \end{aligned} \quad (6.7)$$

Let  $\Delta^k$  denote the operator on  $\mathbb{R}^{\mathbb{N}}$  given by

$$\Delta^k f(i) = k_i^2 (f(i+1) - f(i)) + k_{i-1}^2 (f(i-1) - f(i)), i \in \mathbb{N}, f : \mathbb{N} \rightarrow \mathbb{R}, k_0 = 0, \quad (6.8)$$

and set  $F(i, t) = \mu_r |\partial_i (P_t f)|^2$  for  $t \geq 0, i \in \mathbb{N}$ . Then we can rewrite (6.7) as

$$F(t) \leq F(0) + \int_0^t \Delta^k F(s) ds, \quad t \in [0, \infty). \quad (6.9)$$

Hence, by the positivity of the semigroup  $(e^{t\Delta^k})_{t \geq 0}$ , and Duhamel's principle, we can conclude that

$$F(t) \leq G(t) := e^{t\Delta^k} F(0) \quad (6.10)$$

for  $t \in [0, \infty)$ . It has been shown in [2] that there exist  $C = C(\{k_n\}_{n=1}^\infty) > 0$  such that

$$\sum_i G(i, t) \leq C e^{-\frac{t}{\nu}} \sum_i G(i, 0), t \geq 0. \quad (6.11)$$

where  $\nu$  has been defined in (6.2). Now the result follows from inequalities (6.10) and (6.11). ■

### Corollary 6.2

$$\mu_r (P_t f - \mu_r f)^2 \leq C \mathcal{A}_r(f) e^{-\frac{t}{\nu}}, f \in \mathcal{H}^1.$$

**Proof.** The proof of this result follows immediately from the Poincaré inequality for the Gaussian measure  $\mu_r$ , see, for instance, theorem 5.5.1 , p. 226 in [8]. ■

Define

$$\overline{\mathcal{H}}^1 = \{f \in L^2(\mu_r) \mid \int f d\mu_r = 0, \|f\|_{\overline{\mathcal{H}}^1}^2 = \mathcal{A}_r(f) < \infty\}.$$

Let  $D_{\overline{\mathcal{H}}^1}(\mathcal{L})$  domain of operator  $\mathcal{L}$  in  $\overline{\mathcal{H}}^1$ .

**Corollary 6.3 (Poincaré inequality in  $\overline{\mathcal{H}}^1$ )** *There exists  $C > 0$  such that*

$$\|f\|_{\overline{\mathcal{H}}^1}^2 \leq C < -\mathcal{L}f, f >_{\overline{\mathcal{H}}^1}, f \in D_{\overline{\mathcal{H}}^1}(\mathcal{L}).$$

**Corollary 6.4** *There exists  $C > 0$  such that*

$$\mu_r(f - \mu_r f)^2 \leq 2\nu(-\mathcal{L}f, f)_{L^2(\mu_r)}(1 + \max(0, \log \frac{C\|f\|_{\overline{\mathcal{H}}^1}^2}{2\nu(-\mathcal{L}f, f)_{L^2(\mu_r)}})), f \in D(\mathcal{L}) \cap \mathcal{H}^1.$$

Corollaries 6.3 and 6.4 can be deduced from the theorem 6.1 in the same way as Nash-Liggett inequalities has been proven in [15], see proof of Theorem 8.1.

**Remark 6.5** *The convergence in Theorem 6.1 cannot be improved. Indeed, let  $S(l, t) = P_t(x_l^2)$  for  $t \geq 0$  and  $l \in \mathbb{N}$ . Then  $\mathcal{L}x_l^2 = k_l^2(x_{l+1}^2 - x_l^2) + k_{l-1}^2(x_{l-1}^2 - x_l^2)$ ,  $l \in \mathbb{N}$ , so that,*

$$\frac{\partial S}{\partial t} = \Delta^k S,$$

where  $\Delta^k$  is defined by formula (6.8). Thus

$$S(t) = e^{t\Delta^k} S(0), t \geq 0, \quad (6.12)$$

so that convergence rate in the Theorem 6.1 is achieved.

**Remark 6.6** *If we consider the system (2.1) with the choice of positive  $\{k_n\}_{n=1}^\infty$  such that  $\sum_{n=1}^\infty \frac{n}{k_n^2} = \infty$  then it is possible to show polynomial rate of convergence to equilibrium for the corresponding semigroup  $P_t, t \geq 0$  in the same way as in the paper [15].*

**Remark 6.7** *We have shown that the exponential rate of convergence for the semigroup  $(P_t)_{t \geq 0}$  holds if  $k_n = \lambda^n, \lambda > 1$ . In the same time, there is no spectral gap if  $\lambda = 1$ . Indeed, it is enough to notice that if  $f_N = \sum_{k=1}^N (x_k^2 - 1)$  then*

$$\|f_N\|_{\overline{\mathcal{H}}^1}^2 \sim N, < -\mathcal{L}f_N, f_N >_{\overline{\mathcal{H}}^1} \text{ is independent upon } N,$$

and corollary 13 does not hold. Thus, the asymptotic behaviour of our conservative system depends upon the value of the parameter  $\lambda$ .

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